J. Fluid Mech. (1990), *vol.* 216, *pp.* 193–212
Printed in Great Britain

Sound generation by a supersonic aerofoil cutting through a steady jet flow

By Y. P. GUO
St John's College, Cambridge CB2 1TP, UK

(Received 14 June 1989)

This paper examines the sound generation process when a supersonic aerofoil cuts through a steady jet flow. It is shown that the principal sound is generated by the leading edge of the aerofoil when it interacts with the streaming jet. To the leading order in terms of the jet velocity, no trailing-edge sound is generated. This is not the result of the cancellation of a trailing-edge sound by that from vortex shedding through the imposition of the Kutta condition. Instead, the null acoustic radiation from the trailing edge is entirely because, to the leading order, there is no interaction between the trailing edge and the jet. The effect of the trailing edge is to diffract sound waves generated by the leading edge. It is shown that the diffracted field (as well as the incident field) is regular at the trailing edge and the issue of satisfying the Kutta condition does not arise during the diffraction process. Thus, there is no extra vortex shedding from the trailing edge owing to its interaction with the flow, apart from those resulting from the discontinuity across the aerofoil, generated by the flow–leading edge interaction. This is in sharp contrast to the case of subsonic aerofoils where the removal of the singularity in the diffracted field at the trailing edge through the imposition of the Kutta condition results in vortex shedding from the sharp edge and energy exchange between the sound field and the vortical wake.

1. Introduction

In a recent paper (Ffowcs Williams & Guo 1988), we examined the sound generated from the interruption of a steady jet flow by a supersonically moving aerofoil, to model the noise from the interactions of a propeller blade with vortex flows shed from other blades in a high-speed multistage contrarotating propeller system where the vortex core flows have velocity defects from the mean convecting flow. In that study, the supersonic aerofoil was assumed to be semi-infinite, on the grounds that the sound of the leading edge would not be overtaken by that, if there is any, from the trailing edge of a propeller blade of finite chord in the most noisy Mach wave direction, so that the two edges can be treated separately as the leading and trailing edge of two semi-infinite aerofoils. Though this model reveals the basic features of the sound from the flow–leading edge interaction, it is of interest to see whether the trailing edge can cause any appreciable acoustic radiation and whether the sound from the leading edge is the dominant component. Also, it is important to examine the way by which the trailing edge diffracts the sound waves from the leading edge because the diffraction may alter the sound energy radiated to infinity. The case of sound diffraction by a subsonic trailing edge is a typical example where acoustic energy is converted into vortical energy at the edge through vortex *shedding* (e.g. Howe 1980; Crighton 1981; Rienstra 1981). To bring out the effects of the supersonic trailing edge on the sound radiation process, we consider in this paper the sound generated when a supersonically moving aerofoil of finite chord cuts through a steady jet flow. In doing so, we find that the supersonic trailing edge plays a quite different role from that of a subsonic one.

We choose to work with the canonical model of a steady jet because it is simple enough to be analysed exactly, but yet contains the essential features of the complete vortex flow–aerofoil interaction problem. For efficient supersonic aerofoils working at low angles of incidence and conforming with linearized inviscid supersonic aerodynamics, the problem of sound generation can be formulated in a very simple form. The sound can be conveniently expressed, with nonlinear quadrupole sources neglected, in terms of surface pressure fluctuations on the aerofoil (Ffowcs Williams & Hawkings 1969), which, in the case of supersonic aerofoils, can be easily determined by a semi-infinite aerofoil model. This is because disturbances produced by the trailing edge are all confined to the region behind it and pressures on the aerofoil surface are all contributions from the leading edge alone, a remarkable feature of supersonic problems that has in the past been widely utilized in studies of unsteady loading on supersonic wings (e.g. Miles 1959).

The dominant source of acoustic radiation is identified to be the leading edge of the aerofoil, which interrupts the streaming jet, imposing a sudden change of boundary constraint on the jet flow which results in the radiation of compressive waves. These waves propagate to the far field as the principal sound in the form of a pressure pulse with a sharp peak produced when the leading edge is near the centre of the jet flow where the interaction is strongest. This principal sound has characteristics similar to that produced by a semi-infinite aerofoil (Ffowcs Williams & Guo 1988), but now the primary pulse is followed by a secondary pulse of opposite sign which has much smaller amplitude and a much longer duration than the primary pulse. This secondary pulse is due to the diffraction, at the trailing edge, of the sound waves generated by the flow–leading edge interaction.

The diffracted field produced by the supersonic trailing edge has distinctly different features from that in the case of subsonic aerofoils, where (Amiet 1986*a*, *b*) the diffracted field is a single peaked pulse, because only one group of waves that travel in the direction opposite to the aerofoil motion is diffracted by the subsonic trailing edge, and the diffraction has a long tail. The far-field pressure fluctuations return to zero as time tends to infinity, since sound waves travel faster than the subsonic aerofoil so that there are an infinite number of multiple diffractions between the leading and the trailing edge of the aerofoil. For supersonic aerofoils, the trailing edge moves faster than sound so that it diffracts not only the waves travelling opposite to the aerofoil motion but also those that propagate in the same direction as the aerofoil, and thus produces two peaks in the far-field pressure. The supersonic aerofoil moves faster than the disturbances produced by it so that it overtakes all the waves within a finite time. After this, there is no more interaction between the aerofoil and the flow field and the unsteady loading on the aerofoil returns to zero; the far-field sound pressure pulse is consequently of finite duration.

A striking feature of the diffraction by a supersonic trailing edge is that the diffracted field, as well as the incident field from the leading edge, is always regular at the trailing edge owing to the impossibility of the diffracted waves travelling outside the Mach cone emanating from the trailing edge. This is in sharp contrast to the subsonic case and has significant consequences for energy exchange between the acoustic wave field and the vortical wake flow behind the trailing edge. This energy conversion in the case of subsonic aerofoils is effected through vortex shedding, resulting from imposing the Kutta condition at the trailing edge (Crighton 1981, 1985), equivalent to imposing a circular flow around the aerofoil, which removes the singularity in the diffracted field at the trailing edge predicted by the inviscid theory and also determines the unsteady loading on the aerofoil that furnishes the sources of acoustic radiation. This is not the case for supersonic aerofoils because the total flow in this case is always regular, effected through the finite jump across the wedge of the Mach cone. Thus, there is no vorticity production at the supersonic trailing edge due to its interaction with the sound waves; vortex shedding from the aerofoil in this supersonic case results entirely from the asymmetrical wave form from the leading edge. If the incoming waves were symmetrical about the aerofoil plane, the diffraction process would not involve any vortex shedding.

We are also intrigued to find that the trailing edge does not cause any acoustic radiation as it moves through the jet; there is no trace, either in the far field pressure fluctuations or in the unsteady lift on the aerofoil, of the instant at which the trailing edge enters the jet flow. This null acoustic radiation is not due to the cancellation of a trailing-edge sound by that from vortices shed from the trailing edge, as suggested by Howe (1976, 1988) for certain cases of subsonic aerofoils. Instead, it is because, to the leading order in terms of the jet velocity, there is no interaction between the trailing edge and the flow, as similarly analysed by Ffowcs Williams & Guo (1988) for a semi-infinite trailing edge aerofoil and by Amiet (1988) for subsonic aerofoils. Accompanying the generation of the leading-edge sound, a near-field motion is also built up, which is, to the leading order, steady and has the characteristics of incompressible flows. It is this near field that remains with the jet and offsets its velocity to comply with the zero-flow boundary condition through the aerofoil surface. Since both the jet and the induced near field are steady with no pressure fluctuation across the aerofoil, the boundary condition in the wake behind the trailing edge that specified zero pressure jump is satisfied by the flow on its own as the trailing edge approaches the jet. Thus, there is no need for the total flow to adjust itself again when the trailing edge passes through it and no sound is radiated by the passage of the trailing edge. Since there is no interaction between the trailing edge and the jet, the trailing edge does not produce any vorticity. The aerofoil leaves a vortex sheet behind it as it passes through the jet only because its leading edge interrupts the streaming jet, resulting in a discontinuity in the tangential velocity across the aerofoil. The production of this discontinuity in the supersonic case is completely independent of the trailing edge, so that it can be regarded as having no effect on the vorticity production.

2. Formulation

Consider an infinite-span aerofoil of chord $2b$, moving with supersonic speed $cM(M > 1)$ in the positive x_1 direction, c being the constant speed and M the aerofoil Mach number. We choose the Cartesian coordinate system (x_1, x_2, x_3) such that the x_3 axis coincides with the axis of a steady cylindrical flow of radius a and uniform velocity u_0 in the negative x_3 direction (see figure 1). This choice of coordinate system is entirely for the sake of retaining compatibility with our previous study (Ffowcs Williams & Guo 1988); if aerofoil-fixed coordinates were chosen, the same analysis could be conducted and the same results obtained. The aerofoil moves in the plane $x_3 = 0$. At time $t = 0$, the leading edge of the aerofoil reaches the centre of the cylindrical flow. We assume that the velocity of the jet flow and the induced

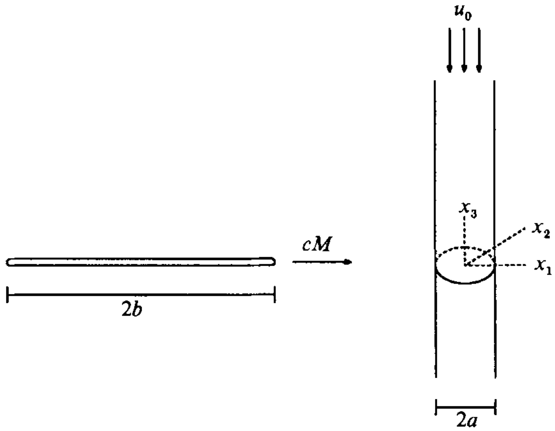

FIGURE 1. The geometry and coordinates of the model problem.

disturbances are of small amplitude so that the use of linear theory is justified. Owing to the symmetrical geometry, it is sufficient to consider only the region $x_3 \geqslant 0$, in which the pressure fluctuations $p(\boldsymbol{x}, t)$ comply with the wave equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0. \tag{2.1}$$

Since the aerofoil moves supersonically, the perturbation pressure in the plane $x_3 = 0$, in which the aerofoil moves, is zero everywhere except on the aerofoil surface; it vanishes ahead of the aerofoil because no disturbance can travel faster than the leading edge that advances supersonically and it is zero behind the aerofoil owing to the symmetrical geometry. Thus, if $p_s(x_\alpha, t)$ denotes the surface pressure on the aerofoil, x_α being the horizontal coordinates ($\alpha = 1, 2$), the boundary condition on the plane $x_3 = 0$ can be set as

$$p(x_\alpha, 0, \tau) = p_s(x_\alpha, t)\,[H(x_1 - cMt + 2b) - H(x_1 - cMt)], \tag{2.2}$$

where H is the Heaviside step function, equal to one for positive arguments and zero for negative arguments, so that the quantity enclosed in the square bracket is equal to one on the aerofoil surface and vanishes elsewhere.

The problem formulated by (2.1) and (2.2) can be solved by taking Fourier transformations from the horizontal coordinate x_α and time t to the wavenumber k_α and frequency ω according to

$$p(x_\alpha, x_3, t) = \frac{cM}{(2\pi)^3}\int_{k_\alpha}\int_{k_0} \hat{p}(k_\alpha, x_3, k_0)\,\mathrm{e}^{-\mathrm{i}(k_\alpha x_\alpha + k_0 cMt)}\,\mathrm{d}^2k_\alpha\,\mathrm{d}k_0,$$

and

$$\hat{p}(k_\alpha, x_3, k_0) = \int_{x_\alpha}\int_t p(x_\alpha, x_3, t)\,\mathrm{e}^{\mathrm{i}(k_\alpha x_\alpha + k_0 cMt)}\,\mathrm{d}^2x_\alpha\,\mathrm{d}t,$$

where the symbol $\hat{}$ stands for quantities in the wavenumber–frequency space and we have denoted the frequency parameter $k_0 = \omega/cM$. Applying this to (2.1), the solution that satisfies the radiation condition at $x_3 \rightarrow +\infty$ is simply

$$\hat{p}(k_\alpha, x_3, k_0) = C(k_1, k_2, k_0)\,\mathrm{e}^{\mathrm{i}\gamma x_3}, \tag{2.3}$$

where C is to be determined from the boundary condition (2.2) and the function $\gamma(k_1, k_2, k_0)$ is defined by

$$\gamma(k_1,k_2,k_0) = \begin{cases} \operatorname{sgn}(k_0)\,(M^2k_0^2-k_1^2-k_2^2)^{\frac{1}{2}} & \text{when } \gamma^2 = M^2k_0^2-k_1^2-k_2^2 \geqslant 0, \quad (2.4a) \\ \mathrm{i}(k_1^2+k_2^2-M^2k_0^2)^{\frac{1}{2}} & \text{when } \gamma^2 = M^2k_0^2-k_1^2-k_2^2 \leqslant 0, \quad (2.4b) \end{cases}$$

with $\operatorname{sgn}(k_0)$ denoting the sign of k_0. This specification of γ ensures that the induced disturbances are either outgoing or finite at infinity.

To determine $C(k_1, k_2, k_0)$, the pressure on the aerofoil surface must be obtained. This can be done by noticing that, since the aerofoil moves supersonically, any disturbances generated by the trailing edge are always confined to the region behind the aerofoil; the pressure fluctuations on the aerofoil are produced by the leading edge alone and are unaware of the existence of the trailing edge. Thus, $p_s(x_\alpha, t)$ is identical to the pressure that would be produced if the aerofoil were semi-infinite. This is in fact the technique used to determine the unsteady wing loading for supersonic aerofoils, which has in the past been thoroughly studied (see, for example, Miles 1959). For a semi-infinite aerofoil, the surface pressure on the aerofoil can be found in a straightforward way by specifying the boundary condition on the plane $x_3 = 0$ in terms of the normal velocity fluctuations, and by either utilizing the Kirchhoff theorem, as was done by Ffowcs Williams & Guo (1988), or using the method of Fourier transformations. The latter is more convenient for the purpose of this paper and is briefly described in the Appendix, in which it is found that

$$p_s(x_\alpha, t) = \frac{\mathrm{i}\rho_0 u_0 a}{(2\pi)^2} \int_{k_\alpha} \int_{k_0} \frac{J_1(a\lambda)}{\lambda} \frac{1}{\gamma(k_1, k_2, k_0)} \mathrm{e}^{-\mathrm{i}(k_\alpha x_\alpha + k_0 cMt)}\, \mathrm{d}^2k_\alpha\, \mathrm{d}k_0, \tag{2.5}$$

where ρ_0 is the constant mean density, J_1 denotes the Bessel function of first order and $\lambda = ((k_1+k_0)^2+k_2^2)^{\frac{1}{2}}$.

On substituting (2.5) into (2.2), taking the Fourier transform and comparing the result with (2.3), it follows that C can be determined as

$$C(k_1, k_2, k_0) = \rho_0 u_0 a \frac{J_1(a\lambda)}{\lambda} \int_{k_0'} \frac{1}{\gamma(k_1^*, k_2, k_0')} \frac{1-\mathrm{e}^{-2\mathrm{i}b(k_0'-k_0)}}{k_0'-k_0} \mathrm{d}k_0', \tag{2.6}$$

where $\gamma(k_1^*, k_2, k_0')$ is given by (2.4) with k_0 replaced by k_0' and k_1 replaced by $k_1^* = k_1 + k_0 - k_0'$. It can be noted that the k_0' integral in (2.6) is in fact the inverse Fourier transform with respect to frequency. As analysed in the Appendix, the integration path for this integral should run above all singularities in the integrand and the contour should be closed by a semicircle at infinity in the lower half complex k_0' plane, on which the contributions to (2.6) vanish. This is required in order to satisfy the causality condition that there is no disturbance at the observation point $\boldsymbol{x}$ until the first generated wave arrives. Since the only singularities are the branch points, determined by $\gamma(k_1^*, k_2, k_0') = 0$, with the branch cut joining them (the reason for this choice of the branch cut is given in the Appendix), the integral can be deformed onto the branch cut, on which $\gamma^2(k_1^*, k_2, k_0') \leqslant 0$, so that γ is given by (2.4b). Thus, we have

$$C(k_1, k_2, k_0) = -2\mathrm{i}\rho_0 u_0 a \frac{J_1(a\lambda)}{\lambda} \int_{k_0'} \frac{H(k_1^{*2}+k_2^2-M^2k_0'^2)}{(k_1^{*2}+k_2^2-M^2k_0'^2)^{\frac{1}{2}}} \frac{1-\mathrm{e}^{2\mathrm{i}b(k_0'-k_0)}}{k_0'-k_0} \mathrm{d}k_0', \tag{2.7}$$

where the factor 2 accounts for the fact that contributions from both sides of the branch cut are equal and the integral is now an ordinary real variable integral with the Heaviside function in the integrand setting the bounds of integration.

With C given by (2.7), the inverse Fourier transform of (2.3) yields

$$p(\boldsymbol{x},t)=\frac{cM}{(2\pi)^3}\int_{k_\alpha}\int_{k_0}C(k_1,k_2,k_0)\,\mathrm{e}^{\mathrm{i}\gamma x_3}\,\mathrm{e}^{-\mathrm{i}(k_\alpha x_\alpha+k_0 cMt)}\,\mathrm{d}^2k_\alpha\,\mathrm{d}k_0, \tag{2.8}$$

where, again, the k_0 integral, being the inverse Fourier transform with respect to frequency, is along a horizontal path in the complex k_0 plane that is above all the singularities in the integrand, so that the causality condition is satisfied. This formulation gives the sound field in terms of the surface pressure fluctuations on the aerofoil which, in the case of supersonic aerofoils, are uniquely determined by the contributions from the leading edge alone. This is actually the Ffowcs Williams & Hawkings (1969) formation of the Lighthill (1952) theory applied to a plane boundary, with the nonlinear quadrupole sources neglected, which are of the order u_0^2, one order of magnitude smaller than (2.8).

3. The sound in the far field

To obtain pressures in the far field, it is convenient to introduce the spherical coordinate system $(|\boldsymbol{x}|,\theta,\phi)$ defined by $x_1=|\boldsymbol{x}|\sin\theta\cos\phi$, $x_2=|\boldsymbol{x}|\sin\theta\sin\phi$ and $x_3=|\boldsymbol{x}|\cos\theta$, in terms of which the k_α integral in the result (2.8) can be written as

$$\int_{k_\alpha}C(k_1,k_2,k_0)\,\mathrm{e}^{\mathrm{i}|\boldsymbol{x}|\psi(k_\alpha)\,\mathrm{d}^2k_\alpha}, \tag{3.1}$$

where $\psi(k_\alpha)$ stands for the phase function

$$\psi(k_\alpha)=\gamma\cos\theta-k_1\sin\theta\cos\phi-k_2\sin\theta\sin\phi.$$

In the far field $|\boldsymbol{x}|\to\infty$, the double integral in (3.1) can be evaluated by the method of two-dimensional stationary phase (e.g. Jones 1972; Lighthill 1978), with the leading-order contribution given by

$$\frac{-2\pi\mathrm{i}k_0M\cos^2\theta}{|\boldsymbol{x}|\alpha}C(\tilde{k}_1,\tilde{k}_2,k_0)\,\mathrm{e}^{\mathrm{i}|\boldsymbol{x}|k_0M}, \tag{3.2}$$

where $\tilde{k}_\alpha$ is the values of k_α at the stationary point determined by the vanishing of the gradient $\boldsymbol{\nabla}\psi(k_\alpha)$, which gives

$$\left.\begin{aligned}\tilde{k}_1&=-Mk_0\sin\theta\cos\phi,\\ \tilde{k}_2&=-Mk_0\sin\theta\sin\phi.\end{aligned}\right\} \tag{3.3}$$

On substituting (3.2) into (2.8), with $C(\tilde{k}_1,\tilde{k}_2,k_0)$ calculated from (2.7), the sound in the far field is found to be

$$p(\boldsymbol{x},t)=-\frac{\rho_0au_0cM^2\cos^2\theta}{2\pi^2|\boldsymbol{x}|\alpha\beta}\int_{k_0}\int_{k_0'}J_1(a\beta k_0)\frac{H(\tilde{k}_1^{*2}+\tilde{k}_2^2-M^2k_0'^2)}{(\tilde{k}_1^{*2}+\tilde{k}_2^2-M^2k_0'^2)^{\frac{1}{2}}}\frac{1-\mathrm{e}^{-2\mathrm{i}b(k_0'-k_0)}}{k_0'-k_0}\times\mathrm{e}^{-\mathrm{i}k_0\tau}\,\mathrm{d}k_0\,\mathrm{d}k_0', \tag{3.4}$$

where τ is the retarded observation position $M(ct-|\boldsymbol{x}|)$ and both α and β are positive quantities introduced to simplify the expression:

$$\alpha=[(1-\sin^2\theta\cos^2\phi)(1-\sin^2\theta\sin^2\phi)]^{\frac{1}{2}}.$$
$$\beta=(1+M^2\sin^2\theta-2M\sin\theta\cos\phi)^{\frac{1}{2}}.$$

With $\tilde{k}_1^* = \tilde{k}_1 + k_0 - k_0'$ and $\tilde{k}_\alpha$ given by (3.3), it can be seen that the integrand in (3.4) is homogenous in k_0 and k_0', so that it is convenient to replace the k_0' integral by η through the change of variables $\eta = (k_0' - k_0)/k_0$, with which the result (3.4) can be simplified to

$$p(\boldsymbol{x}, t) = -\frac{\rho_0 a u_0 c M^2 \cos^2\theta}{2\pi^2 |\boldsymbol{x}| \alpha\beta} \int_{k_0} \int_{\eta} \frac{J_1(a\beta k_0)}{k_0} \frac{H(\mu^2)}{\mu(\eta)} \frac{1 - \mathrm{e}^{-2\mathrm{i}bk_0\eta}}{\eta} \mathrm{e}^{-\mathrm{i}k_0\tau} \,\mathrm{d}k_0 \,\mathrm{d}\eta, \quad (3.5)$$

where $\mu(\eta)$ is a function of η alone and results from inserting $\eta = (k_0' - k_0)/k_0$ into the square root in (3.4) and dividing the result by $|k_0|$, that is

$$\mu = [(1 - M^2)\eta^2 - 2M\eta(M - \sin\theta\cos\phi) - M^2\cos^2\theta]^{\frac{1}{2}}.$$

Now, the k_0 integral in (3.5) can be performed immediately to give

$$\int_{-\infty}^{+\infty} \frac{J_1(a\beta k_0)}{k_0} (1 - \mathrm{e}^{-2\mathrm{i}bk_0\eta}) \,\mathrm{e}^{-\mathrm{i}k_0\tau} \,\mathrm{d}k_0 = \frac{2}{a\beta} [G_\ell(\tau) - G_\ell(\tau + 2b\eta)], \quad (3.6)$$

where G_ℓ is defined by

$$G_\ell(\tau) = \int_{-\infty}^{+\infty} \frac{J_1(a\beta k_0)}{k_0} \mathrm{e}^{-\mathrm{i}k_0\tau} \,\mathrm{d}k_0 = H(a\beta - |\tau|)\,(a^2\beta^2 - \tau^2)^{\frac{1}{2}},$$

the last step of which follows from the formula (6.693) given by Gradshteyn & Ryzhik (1980).

On substituting (3.6) into (3.5), the η-integral in the first term can be calculated immediately with the result $-\pi/M\cos\theta$, because $G_\ell(\tau)$ is independent of η. Denoting the second term by

$$G_\mathrm{d} = \frac{M\cos\theta}{\pi} \int_\eta \frac{[a^2\beta^2 - (\tau + 2b\eta)^2]^{\frac{1}{2}}}{\eta[(1 - M^2)\eta^2 - 2M\eta(M - \sin\theta\cos\phi) - M^2\cos^2\theta]^{\frac{1}{2}}} \,\mathrm{d}\eta, \quad (3.7)$$

The far-field pressure can be written as

$$p(\boldsymbol{x}, t) = \frac{\rho_0 u_0 c M \cos\theta}{\pi |\boldsymbol{x}| \alpha\beta^2} [G_\ell(\tau) + G_\mathrm{d}(\tau)], \quad (3.8)$$

where G_ℓ and G_d respectively characterize the effects of the leading and the trailing edge, and the integration limits in (3.7) are jointly determined by $\mu^2 \geqslant 0$ and $a\beta - |\tau + 2b\eta| \geqslant 0$. The solutions of these two inequalities show that G_d assumes different forms in five regions in the axis of the retarded observation position τ, the dividing points of which are given by

$$\tau_j = \frac{2bM}{M^2 - 1} [M - \sin\theta\cos\phi + \operatorname{sgn}(j - 2.5)\,\Delta] + \cos(j\pi)\,a\beta, \quad j = 1, 2, 3, 4,$$

where Δ is defined by $\Delta = [(M\sin\theta - \cos\phi)^2 + \cos^2\theta\sin^2\phi]^{\frac{1}{2}}$ and

$$\tau_1 < \tau_2 < \tau_3 < \tau_4 \quad \text{if} \quad a < \frac{2bM\Delta}{\beta(M^2 - 1)},$$

$$\tau_1 < \tau_3 < \tau_2 < \tau_4 \quad \text{if} \quad a > \frac{2bM\Delta}{\beta(M^2 - 1)}.$$

These dividing points show the time sequence of the diffraction process when the trailing edge of the aerofoil interacts with the waves produced by the leading edge. From their definitions, it is clear that τ_1 and τ_2 respectively correspond to the instants when the trailing edge enters and leaves the group of waves from the leading edge travelling in the direction opposite to the aerofoil motion, and τ_3 and τ_4 correspond to the instants when it enters and leaves the group travelling in the same direction as the aerofoil. For compact vortex flows with $a < 2bM\Delta/\beta(M^2-1)$, the interaction between the leading edge and the jet flow ends well before the trailing edge encounters the waves generated by that interaction, so that the two groups of waves travelling in opposite directions are separated from each other when they are diffracted by the trailing edge; the trailing edge diffracts all the waves travelling opposite to the aerofoil motion before it catches up with those in the aerofoil motion direction ($\tau_2 < \tau_3$). On the other hand, for large jet radius with $a > 2bM\Delta/\beta(M^2-1)$, the two groups of waves have tails that are mixed with each other. Thus, the trailing edge enters the group of waves propagating in the aerofoil motion direction before it leaves the other group, so that $\tau_3 < \tau_2$.

Having determined the integration bounds, (3.7) can be calculated by making use of the formulae (3.147) to (3.149) given by Gradshteyn & Ryzhik (1980) with the result expressed as a piecewise-smooth function of the retarded observation position $\tau = M(ct-|\boldsymbol{x}|)$. The solution can be facilitated by introducing the following quantities:

$$n = \frac{\tau-\tau_1}{2a\beta}, \quad n' = n\frac{\tau-a\beta}{\tau_2-a\beta},$$

$$m = (\tau_4-\tau)\frac{M^2-1}{4bM\Delta}, \quad m' = m\frac{\tau_2-a\beta}{\tau-a\beta},$$

$$\varpi = \frac{\tau_1-\tau}{\tau_2-\tau}\frac{n-n'}{n}, \quad \varpi' = \frac{\tau+a\beta}{\tau_1+a\beta},$$

and

$$q = \frac{1}{2}\left(\frac{M^2-1}{2bMa\beta\Delta}\right)^{\frac{1}{2}}[(\tau-\tau_1)(\tau_4-\tau)]^{\frac{1}{2}},$$

in terms of which G_d can be written as

$$G_\mathrm{d} = \frac{(\tau_2-\tau)\,q}{[(\tau-\tau_1)(\tau_4-\tau)]^{\frac{1}{2}}}\begin{cases} 0, & -\infty \leqslant \tau \leqslant \tau_1, \\ \Pi\left(\frac{\pi}{2},n,q\right)-\varpi'\Pi\left(\frac{\pi}{2},n',q\right), & \tau_1 \leqslant \tau \leqslant \tau_2, \\ \frac{1}{q}\left[\varpi F\left(\frac{\pi}{2},\frac{1}{q}\right)-\Pi\left(\frac{\pi}{2},\frac{1}{n},\frac{1}{q}\right)+\varpi'\Pi\left(\frac{\pi}{2},\frac{1}{n'},\frac{1}{q}\right)\right], & \tau_2 \leqslant \tau \leqslant \tau_3, \\ \varpi F\left(\frac{\pi}{2},q\right)-\Pi\left(\frac{\pi}{2},m,q\right)+\varpi'\Pi\left(\frac{\pi}{2},m',q\right), & \tau_3 \leqslant \tau \leqslant \tau_4, \\ 0, & \tau_4 \leqslant \tau \leqslant +\infty, \end{cases}$$

for $a \leqslant 2bM\Delta/\beta(M^2-1)$, where F and Π are respectively the elliptic functions of first and third kind. For the case of large jet radius with $a \geqslant 2bM\Delta/\beta(M^2-1)$, the solution can be obtained from the above result by exchanging τ_2 and τ_3 and then replacing the expression for the region $\tau_3 \leqslant \tau \leqslant \tau_2$ by

$$\frac{\tau_2-\tau}{[(\tau-\tau_1)(\tau_4-\tau)]^{\frac{1}{2}}}\left[\Pi\left(\frac{\pi}{2},\frac{1}{m},\frac{1}{q}\right)+\varpi'\Pi\left(\frac{\pi}{2},\frac{1}{m'},\frac{1}{q}\right)\right].$$

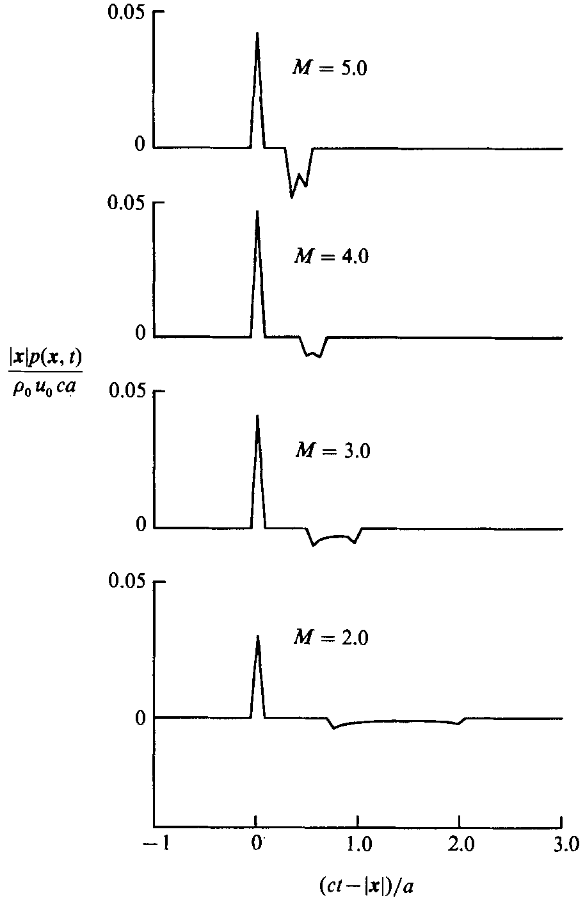

FIGURE 2. The far-field sound pressure with $a/b = 0.1$ and $\theta = 0$.

Typical far-field pressure waveforms, calculated according to (3.8) for some values of the aerofoil Mach number M, are plotted in figure 2 as a function of the retarded observation position $ct-|\boldsymbol{x}|$. The dominant sound is generated in the form of a positive pulse when the leading edge interacts with the jet flow. The sharp peak of the pulse is located near $\tau = 0$, indicating that it is produced when the leading edge is near the centre of the streaming jet where the interaction is strongest. This positive pulse has characteristics similar to that produced by a semi-infinite aerofoil; it spreads spherically with decaying amplitude according to $1/|\boldsymbol{x}|$, except in the Mach wave direction where its amplitude remains constant as the pulse travels away, as analysed by Ffowcs Williams & Guo (1988). The pulse from the flow–leading edge interaction is followed by a secondary pulse of negative sign, which has much smaller amplitude and a much longer duration. The area under the positive pulse is always equal to that under the negative one, which is clear from (3.5), the integration of which with respect to τ is identically zero. The secondary pulse is switched on at $\tau = \tau_1$ and off at $\tau = \tau_4$; it is the diffracted field produced when the trailing edge moves through the waves from the leading edge.

The pulse due to diffraction has two peaks and a finite duration, which is different from that produced by a subsonic aerofoil. For a subsonic aerofoil (e.g. Amiet 1986*a*, *b*), the waves in the same direction as the aerofoil motion always move ahead of the aerofoil, so that only the waves travelling in the opposite direction to the aerofoil motion are diffracted by the trailing edge, resulting in a single peak in the far-field pressure. In the case of supersonic aerofoils, the trailing edge diffracts both groups as it catches up with them. Since the two diffracted fields are produced separately from each other in time, the resultant far-field sound contains two distinct peaks in the pressure waveform. In the subsonic case, the secondary pulse has a long tail; the pressure returns to zero as $\tau \rightarrow +\infty$, because the diffracted field can propagate ahead of the trailing edge that produced it. Thus, the trailing edge always moves in an unsteady wave field and there are an infinite number of multiple diffractions between the trailing and the leading edge, though the strength of the diffraction rapidly becomes very weak. On the other hand, a wave from the supersonic leading edge will be diffracted by the trailing edge only once because the supersonic aerofoil overtakes all the disturbances within a finite time, after which there is no more interaction between the aerofoil and the total flow field and the unsteady loading on the aerofoil vanishes (see the next section). The pressure in the far field then returns to zero after a finite time.

From the result (3.8), it is clear that the far-field sound contains no trace of the interaction between the trailing edge and the streaming jet flow; no sound is generated by the trailing edge as it passes through the jet. The null acoustic radiation from the trailing edge is not the result of the cancellation of trailing-edge sound by that from the vortex shedding process effected by the imposition of the Kutta condition at the sharp edge, as in some subsonic cases (Howe 1976, 1988). Instead, it is entirely due to the fact that, to the leading order in terms of the jet velocity, there is no interaction between the flow and the trailing edge. This is similar to the situation analysed by Amiet (1988) for subsonic aerofoils and by Ffowcs Williams & Guo (1988) for supersonic semi-infinite trailing-edge aerofoils. It can be understood by noticing that, as the leading edge penetrates the jet flow, it not only scatters acoustic waves but also builds up a near-field motion which is essentially steady and with the characteristics of incompressible flows. It is this near-field motion that remains with the jet flow and offsets its velocity to satisfy the zero-flow boundary condition through the aerofoil surface. Since both the induced near field and the streaming jet are steady with no pressure jump across the aerofoil, the boundary condition behind the trailing edge that specifies zero pressure jump is automatically satisfied by the flow on its own as the trailing edge approaches the jet. Hence, there is no need for the flow to change as the trailing edge enters it and no sound is radiated.

Though both the linear theory described above (and that given by Amiet for subsonic aerofoils) and the acoustic analogy developed by Howe (1976, 1988) predict null acoustic radiation from the trailing edge, the mechanisms involved are quite different. Most noticeably, since there is no interaction between the trailing edge and the jet flow in linear theory, there is no vorticity production at the trailing edge as it moves through the jet, which is essential in Howe's cancellation mechanism. This conclusion is based on linear theory where no unsteady near-field pressure is induced by the flow–leading edge interaction. If nonlinear effects were included, there would be a trailing-edge radiation; the flow–leading edge interaction would induce an unsteady near-field pressure in the vicinity of the steady jet, which would be scattered into sound by the trailing edge as it moves through the jet flow. However, as analysed by Amiet (1988) for subsonic aerofoils, the sound from this mechanism would be of negligible importance in comparison with that from the leading edge, if the same jet were to interact with both the edges. This is because the unsteady pressure fluctuations due to nonlinear effects, which would furnish the strength of the trailing-edge noise, should be proportional to the square of the jet velocity, an order of magnitude smaller than the dominant sound from the flow–leading edge interaction which is characterized by a pressure field in proportion to the jet velocity.

4. The unsteady loading on the aerofoil

Owing to the radiation of compressive waves, the aerofoil experiences an unsteady lift during the interaction. This lift can be calculated from a semi-infinite aerofoil model because the surface pressure on the supersonic aerofoil is determined by the leading edge alone (Miles 1959), which assumes the form

$$L(t) = 2\int_{-\infty}^{+\infty}\int_{cMt-2b}^{cMt} p_s(x_\alpha, t)\,\mathrm{d}^2x_\alpha, \tag{4.1}$$

where the factor 2 accounts for the contribution from the lower surface of the aerofoil and $p_s(x_\alpha, t)$ is the surface pressure given by (2.5). On substituting (2.5) into (4.1), the x_2 integral can be performed trivially as $2\pi\delta(k_2)$, which in turn can be utilized to carry out the k_2 integral. The integration with respect to x_1 can also be explicitly evaluated so that we have

$$L(t) = -\frac{\rho_0 u_0 acM}{\pi}\int_{k_1}\int_{k_0}\frac{J_1(a|k_0+k_1|)}{|k_0+k_1|}\frac{1}{k_1\gamma(k_1,0,k_0)}(1-\mathrm{e}^{2\mathrm{i}bk_1})\,\mathrm{e}^{-\mathrm{i}cMt(k_0+k_1)}\,\mathrm{d}k_0\,\mathrm{d}k_1.$$

This result can be rewritten by explicitly specifying the integration limits for the k_0 integral through a Heaviside function, as was done in (2.7). The arguments are similar; the causality condition requires that the integration path for the frequency integral should run above all singularities in the integrand, which are two branch points joined by a cut along the real k_0 axis. On forming a closed contour by a semicircle at infinity in the lower half k_0 plane, the k_0 integral can be deformed onto the branch cut, on which

$$\gamma(k_1, 0, k_0) = \mathrm{i}(k_1^2-M^2k_0^2)^{\frac{1}{2}},$$

so that the lift becomes

$$L(t) = \frac{2\mathrm{i}\rho_0 u_0 acM}{\pi}\int_{k_1}\int_{k_0}\frac{J_1(a|k_0+k_1|)}{|k_0+k_1|}\frac{H(k_1^2-M^2k_0^2)}{k_1(k_1^2-M^2k_0^2)^{\frac{1}{2}}}(1-\mathrm{e}^{2\mathrm{i}bk_1})\,\mathrm{e}^{-\mathrm{i}cMt(k_0+k_1)}\,\mathrm{d}k_0\,\mathrm{d}k_1, \tag{4.2}$$

where the integral is on one side of the cut with contributions from the other side taken into account by the factor 2.

By introducing the change of variables $\zeta = k_0+k_1$ and $\eta = k_1/\zeta$, (4.2) becomes

$$L(t) = \frac{2\mathrm{i}\rho_0 u_0 acM}{\pi}\int_{\zeta}\int_{\eta}\frac{J_1(a\zeta)}{\zeta^2\eta}\frac{H(k_1^2-M^2k_0^2)}{(k_1^2-M^2k_0^2)^{\frac{1}{2}}}(1-\mathrm{e}^{2\mathrm{i}b\zeta\eta})\,\mathrm{e}^{-\mathrm{i}cMt\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\eta, \tag{4.3}$$

in which, the ζ-integral assumes the form

$$-2\mathrm{i}\int_0^{\infty}\frac{J_1(a\zeta)}{\zeta^2}[\sin(\zeta cMt)-\sin\zeta(cMt-2b\eta)]\,\mathrm{d}\zeta = -\mathrm{i}a[Q_\ell(cMt)-Q_\ell(cMt-2b\eta)], \tag{4.4}$$

with $Q_\ell(cMt)$ defined by

$$Q_\ell(cMt) = \begin{cases} -\frac{1}{2}\pi, & -\infty \leqslant cMt \leqslant -a, \\ \arcsin\dfrac{cMt}{a} + \dfrac{cMt}{a}\left[1-\dfrac{(cMt)^2}{a^2}\right]^{\frac{1}{2}}, & -a \leqslant cMt \leqslant +a, \\ \frac{1}{2}\pi, & +a \leqslant cMt \leqslant +\infty. \end{cases}$$

On substituting (4.4) into (4.3), it follows that

$$L(t) = \frac{2\rho_0 u_0 cM}{\pi}\left[Q_\ell(cMt)\int_\eta \frac{H[\eta^2-M^2(1-\eta)^2]}{\eta(\eta^2-M^2(1-\eta)^2)^{\frac{1}{2}}}\mathrm{d}\eta \right.$$
$$\left. -\int_\eta \frac{H[\eta^2-M^2(1-\eta)^2]}{\eta[\eta^2-M^2(1-\eta)^2]^{\frac{1}{2}}} Q_\ell(cMt-2b\eta)\,\mathrm{d}\eta\right].$$

The first integral in this result can be calculated immediately with the result π/M and the second can be evaluated in a way similar to that used to calculate (3.7). When this is done, the unsteady lift can be written as

$$L(t) = 2\rho_0 u_0 ca^2[Q_\ell(cMt)-Q_{\mathrm{d}}(cMt)]. \tag{4.5}$$

Here Q_{d} is a piecewise smooth function of time t, given, for the case of small jet radius with $a < 2bM/(M^2-1)$, by

$$2Q_{\mathrm{d}} = \begin{cases} -\pi, & -\infty \leqslant t \leqslant t_1, \\ \arcsin\dfrac{M(cMt+a-2b)}{cMt+a} - \dfrac{\pi}{2} + f\left(\dfrac{M}{M+1}, \dfrac{cMt+a}{2b}\right), & t_1 \leqslant t \leqslant t_2, \\ \arcsin\dfrac{M(cMt+a-2b)}{cMt+a} + \arcsin\dfrac{M(cMt-a-2b)}{cMt-a} & \\ \qquad\qquad + f\left(\dfrac{cMt-a}{2b}, \dfrac{cMt+a}{2b}\right), & t_2 \leqslant t \leqslant t_3, \\ \arcsin\dfrac{M(cMt-a-2b)}{cMt-a} + \dfrac{\pi}{2} + f\left(\dfrac{cMt-a}{2b}, \dfrac{M}{M-1}\right), & t_3 \leqslant t \leqslant t_4, \\ \pi, & t_4 \leqslant t \leqslant +\infty, \end{cases}$$

where $f(\eta_1, \eta_2)$ is a complicated combination of elementary functions and elliptic functions. It can also be expressed in terms of a single integral as

$$f(\eta_1, \eta_2) = \frac{2M}{\pi}\int_{\eta_2}^{\eta_2}\left[\arcsin\frac{cMt-2b\eta}{a} + \frac{cMt-2b\eta}{a}\right.$$
$$\left.\times\left[1-\frac{(cMt-2b\eta)^2}{a^2}\right]^{\frac{1}{2}}\right]\frac{1}{\eta(\eta^2-M^2(1-\eta)^2)^{\frac{1}{2}}}\mathrm{d}\eta.$$

For the case of large jet radius with $a > 2bM(M^2-1)$, the solution can be obtained by simply exchanging t_2 and t_3 and then replacing the result in the region $t_3 \leqslant t \leqslant t_2$ by

$$f\left(\frac{M}{M+1}, \frac{M}{M-1}\right),$$

where t_j is given by

$$t_j = \frac{2b/c}{M+\mathrm{sgn}\,(2.5-j)} + \cos(\pi j)\frac{a}{cM}, \quad j = 1, 2, 3, 4.$$

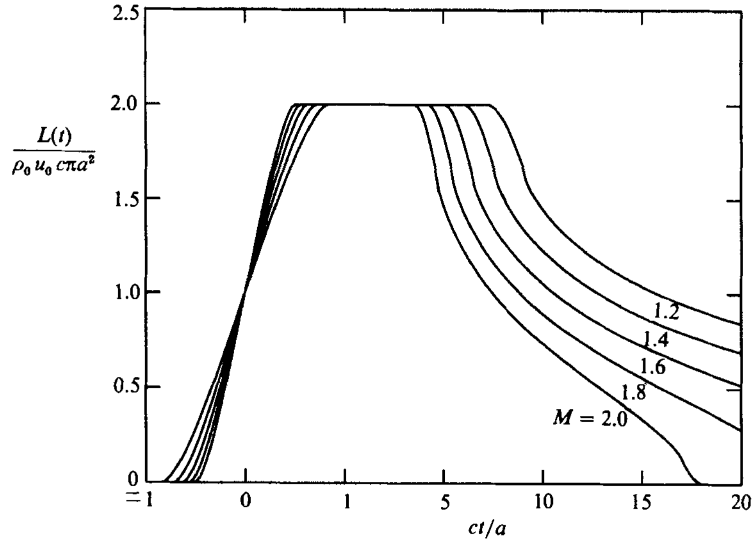

FIGURE 3. The unsteady lift on the aerofoil with $a/b = 0.1$.

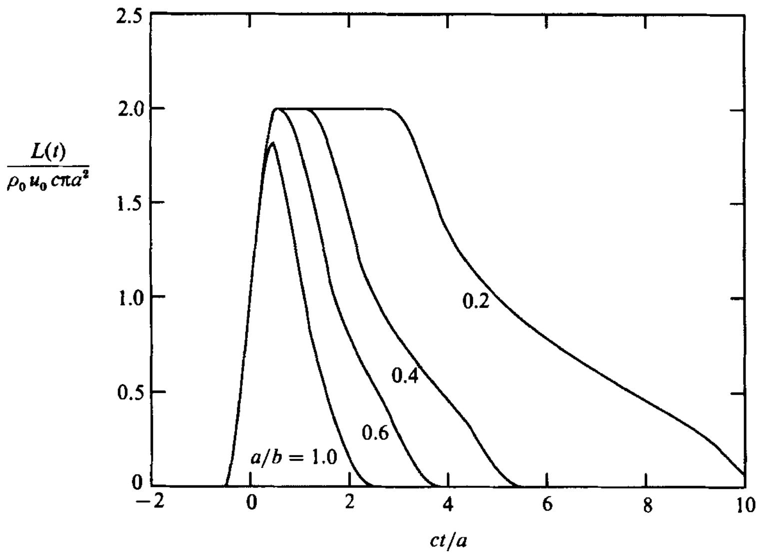

FIGURE 4. The unsteady lift on the aerofoil with $M = 2.0$.

The unsteady lift on the aerofoil is plotted in figures 3 and 4 as a function of time t, showing the dependence of the lift on the aerofoil Mach number M and the ratio of the jet radius a to the aerofoil semichord b. The lift is initially zero until the leading edge reaches the jet flow at time $t = -a/cM$, after which it grows with time continuously as the leading edge penetrates the jet flow. The growth ceases at a later time when the lift reaches a constant value, or when it abruptly begins to decay, depending on the ratio a/b. The gradual initial growth of the lift is due to the finite spatial dimension of the jet flow that is interrupted by the aerofoil; if the velocity of the jet had a delta-function distribution in space, the lift would suddenly grow from zero to a constant value at the instant when the leading edge strikes the jet. The initial growth is independent of the aerofoil chord but becomes more rapid as the aerofoil Mach number increases. At $t = 0$ when the aerofoil has blocked half the jet flow, the lift is the same for any Mach number.

Figure 3 shows the unsteady lift for the case of compact jet radius ($a/b = 0.1$). In this case, the interaction between the leading edge and the streaming jet occurs before any of the waves generated by that interaction encounters the trailing edge, so that the lift reaches the constant value $2\rho_0 u_0 c\pi a^2$ before it starts to decay. This value can be given a physical interpretation; it is the lift that would be experienced by a semi-infinite supersonic aerofoil cutting through a delta-function jet flow of strength $u_0 \pi a^2$, namely of the same volume flux as the cylindrical jet. This is because, for this part of the lift curve, the aerofoil has completely blocked the jet but the waves generated by the blockage have not yet reached the trailing edge, the aerofoil behaving as if it were semi-infinite. In fact, this constant value is the maximum lift an aerofoil of arbitrary Mach number and chord may experience; varying the Mach number does not affect this constant value, unless M is reduced to subsonic values, which results in a reduction in the maximum lift (Guo 1989). It can also be noticed that this constant value for the lift is the limit case for subsonic aerofoils as the aerofoil Mach number approaches unity (Amiet 1986*a*, *b*). For our supersonic case, there is no Mach-number dependence on the level of this flat portion of the lift curve because none of the propagating waves extends outside the region of the aerofoil planform until the waves pass the trailing edge; thus, the motion of the aerofoil has no effect on the propagating waves until the trailing edge reaches them. The instant at which the lift begins to decrease is determined by the time within which the first wave generated by the leading edge reaches the trailing edge, that is, t_1. If this happens after the jet is completely blocked by the aerofoil ($t_1 > a/cM$), the lift reaches its constant maximum. On the other hand, if the first wave generated encounters the trailing edge before the leading edge reaches the downstream boundary of the jet flow at $x_1 = a$, which is the case for non-compact jet flows, the lift abruptly starts to decay before reaching the constant value, as shown in figure 4.

5. The field near the trailing edge

To see whether acoustic energy produced by the flow–leading edge interaction is converted into vortical energy at the trailing edge as it moves through and diffracts the waves, it is necessary to examine the diffracted field near the trailing edge. Energy exchange between sound waves and vortices can occur, in potential theory, if the inviscid theory predicts a singular solution at the sharp edge and the singularity is required to be removed by imposing the Kutta condition, resulting in a regular but discontinuous solution which indicates vortex shedding from the trailing edge, a well-known phenomenon for subsonic aerofoils (e.g. Crighton 1985; Howe 1976). This is not the case for supersonic aerofoils because the inviscid solution in this case is regular at the trailing edge and the Kutta condition that requires finite behaviour at the sharp edge is automatically satisfied by the diffracted field. This is demonstrated in this section.

Since the waves produced by the leading edge alone are regular everywhere, it is

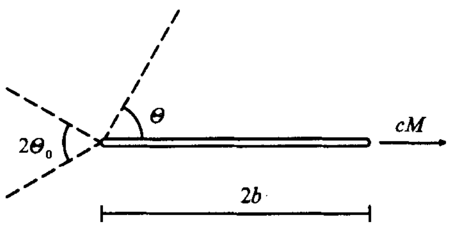

FIGURE 5. The trailing-edge geometry where Θ_0 is the Mach angle.

sufficient to consider only the diffracted field due to the leading edge, which, denoted by $p_d(\boldsymbol{x}, t)$, can be written according to (2.6) and (2.7) as

$$p_d(\boldsymbol{x}, t) = \frac{cM\rho_0 u_0 a}{(2\pi)^3} \int_{k_\alpha} \int_{k_0} \int_{k_0'} \frac{J_1(a\lambda)}{\lambda} \frac{1}{\gamma(k_1^*, k_2, k_0')} \frac{1}{k_0 - k_0'} \times \mathrm{e}^{2\mathrm{i}b(k_0 - k_0')} \, \mathrm{e}^{\mathrm{i}\gamma x_3} \, \mathrm{e}^{-\mathrm{i}(k_\alpha x_\alpha + k_0 cMt)} \, \mathrm{d}^2 k_\alpha \, \mathrm{d}k_0 \, \mathrm{d}k_0'.$$

By changing the k_0 integral from k_0 to $k_0 + k_1$ and introducing the cylindrical coordinates (r, Θ) based on the trailing-edge geometry with $r = [x_3^2 + (x_1 - cMt + 2b)^2]^{\frac{1}{2}}$ so that $x_1 = r\cos\Theta$ and $x_3 = r\sin\Theta$ (see figure 5), the diffracted field $p_d(\boldsymbol{x}, t)$ can be rewritten as

$$p_d(\boldsymbol{x}, t) = \frac{cM\rho_0 u_0 a}{(2\pi)^3} \int_{k_\alpha} \int_{k_0} \int_{k_0'} \frac{J_1(a\lambda')}{\lambda} \frac{1}{\gamma(k_0 - k_0', k_2, k_0')} \frac{1}{k_0 - k_0' - k_1} \times \mathrm{e}^{2\mathrm{i}b(k_0 - k_0')} \, \mathrm{e}^{\mathrm{i}r[\gamma(k_\alpha, k_0 - k_1)\sin\Theta - k_1 \cos\Theta]} \, \mathrm{e}^{-\mathrm{i}(k_2 x_2 + k_0 cMt)} \, \mathrm{d}^2 k_\alpha \, \mathrm{d}k_0 \, \mathrm{d}k_0', \quad (5.1)$$

where $\lambda' = (k_0^2 + k_2^2)^{\frac{1}{2}}$. The result is cast in this form because the dependence of $p_d(\boldsymbol{x}, t)$ on r and Θ is all contained in the k_1 integral which can be readily evaluated in the limit $r \to 0$.

To this end, we note that the path of the k_1 integral is a horizontal line in the complex k_1 plane below all the singularities in the integrand, of which only the branch points determined by $\gamma(k_\alpha, k_0 - k_1) = 0$ are of interest; the pole at $k_1 = k_0 - k_0'$ does not affect the result because its contribution is exactly cancelled by that from a pole in the incoming waves from the leading edge, as is clear from the result (2.6) (so that p_d actually gives the behaviour of the total pressure fluctuations close to the trailing edge). From the definition (2.4), it can be derived that, as $|k_1| \to \infty$,

$$\gamma^2(k_1, k_2, k_0 - k_1) = M^2(k_0 - k_1)^2 - k_\alpha^2 \sim (M^2 - 1)\, k_1^2,$$

which leads to

$$\gamma(k_1, k_2, k_0 - k_1) \sim \operatorname{sgn}(k_0 - k_1)\, |k_1| (M^2 - 1)^{\frac{1}{2}} \sim -k_1 (M^2 - 1)^{\frac{1}{2}}. \quad (5.2)$$

It is then clear that infinity is not a branch point so that the branch cut can be made by joining the two branch points given by $\gamma(k_1, k_2, k_0 - k_1) = 0$. In this case, the k_1 integral can be deformed from its original path onto the branch cut by forming a closed contour with a semicircle at infinity in the upper half complex k_1 plane. The contribution from this semicircle vanishes provided that

$$\operatorname{Im}[\gamma(k_1, k_2, k_0 - k_1)\sin\Theta - k_1 \cos\Theta] > 0 \quad \text{at} \quad |k_1| \to \infty,$$

where Im (z) stands for the imaginary part of z. With $\gamma(k_\alpha, k_0-k_1)$ given by (5.2) and considering that Im $(k_1) > 0$ on the semicircle, this condition leads to

$$(M^2-1)^{\frac{1}{2}}\sin\Theta+\cos\Theta < 0, \tag{5.3}$$

which states that the k_1 integral, and hence the diffracted pressure $p_d(\boldsymbol{x}, t)$, is non-zero only within the Mach cone. Denoting the Mach angle by $\Theta_0 = \arcsin(1/M)$ (see figure 5), the condition (5.3) is equivalent to

$$\Theta+\Theta_0 > \pi.$$

When this condition is satisfied, the k_1 integral in (5.1) is given by that along the branch cut. After some simple algebra, it becomes

$$2H(\Theta+\Theta_0-\pi)\int_{-\infty}^{+\infty}\frac{H[k_\alpha^2-M^2(k_0-k_1)^2}{k_1+k_0'-k_0}\mathrm{e}^{-\mathrm{i}k_1 r\cos\Theta}\sinh\{r\sin\Theta[k_\alpha^2-M^2(k_0-k_1)^2]^{\frac{1}{2}}\}\,\mathrm{d}k_1.$$

Since the integrand is now evaluated at finite values of k_1, it can be expanded in terms of a power series of r. As the trailing edge is approached ($r\rightarrow 0$), the leading-order term in the expansion is of order r and is given by

$$2r\sin\Theta H(\Theta+\Theta_0-\pi)\int_{-\infty}^{+\infty}\frac{[k_\alpha^2-M^2(k_0-k_1)^2]^{\frac{1}{2}}}{k_1+k_0'-k_0}H[k_\alpha^2-M^2(k_0-k_1)^2]\,\mathrm{d}k_1,$$

which can be readily calculated by standard techniques. On substituting this into (5.1), the diffracted pressure $p_d(\boldsymbol{x}, t)$ can be expressed as

$$p_d(\boldsymbol{x}, t) \sim r\sin\Theta H(\Theta+\Theta_0-\pi)B(x_2, t), \tag{5.4}$$

where

$$B(x_2, t) = \frac{2cM\rho_0 u_0 a}{(2\pi)^3}\int_{k_\alpha}\int_{k_0}\int_{k_0'}\frac{J_1(a\lambda')}{\lambda'}\frac{1}{\gamma(k_0-k_0', k_2, k_0')}\frac{[k_\alpha^2-M^2(k_0-k_1)^2]^{\frac{1}{2}}}{k_1+k_0'-k_0}$$
$$\times H[k_\alpha^2-M^2(k_0-k_1)^2]\,\mathrm{e}^{2\mathrm{i}b(k_0-k_0')}\,\mathrm{e}^{-\mathrm{i}(k_2x_2+k_0cMt)}\,\mathrm{d}^2k^\alpha\,\mathrm{d}k_0\,\mathrm{d}k_0'.$$

The result (5.4) clearly reveals the structure of the diffracted pressure field near the trailing edge. It is clear that all the diffracted waves are confined to the region within the Mach cone $\Theta > \pi-\Theta_0$, with maximum wave amplitude in the Mach wave direction (at the wedge of the Mach cone). This maximum is given by $B(x_2, t)$ in (5.4) which is now a function of x_2 and t only. The angular dependence of the waves is characterized by $\sin\Theta$ which is different from the case of subsonic aerofoils where the diffracted field near the sharp edge is proportional to $\sin(\frac{1}{2}\Theta)$ (e.g. Ffowcs Williams & Hall 1970; Crighton 1972; Rienstra 1981).

A more important difference between (5.4) and the diffraction by a subsonic trailing edge is that (5.4) vanishes as the edge is approached; the diffracted field varies linearly with r, in sharp contrast to the case of subsonic aerofoils, where the diffracted field is singular at the edge with an $r^{-\frac{1}{2}}$ singularity. This regular behaviour of the diffracted field near the supersonic trailing edge is similar to that of a finite-strength vortex sheet leaving a supersonic trailing edge (Morgan 1974; Cargill 1982) where the deformation of the vortex sheet is in proportion to the distance from the edge in the vicinity of it. This regular behaviour is significant in that no extra vorticity is produced at the supersonic trailing edge due to its interaction with the compressive waves, as is often the case for subsonic aerofoils; the removal of the singularity at a subsonic trailing edge by imposing the Kutta condition results in vortex shedding and dissipation of acoustic energy. The total flow is always regular at the supersonic trailing edge when it diffracts incoming acoustic waves, because of the impossibility of waves travelling outside the Mach cone, an essential feature of supersonic flows. This feature results in a jump in the diffracted field from zero to a finite value across the wedge of the Mach cone, as characterized by the Heaviside function in (5.4). It is this finite jump, similar to a weak shock that smooths out the large gradient of the diffracted field at the trailing edge, which can only be removed, in subsonic flows, by resorting to the imposition of a circulation around the aerofoil, namely, the Kutta condition. This circulation makes the flow leave the sharp trailing edge tangentially and also determines the lift on the subsonic aerofoil. For supersonic aerofoils, the tangential flow at the trailing edge is achieved by the compressive waves which also determine the unsteady loading on the aerofoil, as demonstrated in the previous section.

6. Discussion and conclusions

The problem of sound generation by a supersonic aerofoil interacting with a steady jet flow is examined in a simple model that allows for exact analytic solutions and reveals fundamental features of the interaction process that causes acoustic radiation. The dominant mechanism is identified to be the flow–leading edge interaction which generates an intense pressure pulse that is heard in the far field as the principal sound. The role played by the supersonic trailing edge during the interaction process is analysed in detail. It is interesting to find that its effects are quite different from those of a subsonic aerofoil. The most striking feature is that there is no extra vortex shedding from the trailing edge due to its interaction with the flow; vortex shedding from the aerofoil in this case is entirely due to the discontinuity across the aerofoil generated by the flow–leading edge interaction. This can also be understood by considering the rate of change of circulation about a path enclosing the aerofoil but not the wake. It is clear that the circulation on such a path does not depend on the conditions at the trailing edge since the edge does not have any upstream influence. Thus, the circulation about the wake, which must be the negative of that on the aerofoil, cannot depend on the trailing-edge condition.

The trailing edge does not cause any acoustic radiation as it moves through the jet flow because there is no interaction between them. The steady near field has a discontinuity in the tangential velocity across the aerofoil, which results in a vortex sheet after the trailing edge has passed the jet, but that vortex shedding is a silent process which does not involve any pressure fluctuations. This analysis is based on linear theory. If nonlinear effects were included, there would be component of sound from the trailing edge; the flow–leading edge interaction would induce an unsteady near-field pressure fluctuation in the vicinity of the steady jet, which would be scattered into sound by the trailing edge when it moves through the jet flow. However, the sound from this mechanism can be shown to be of negligible importance in comparison with that from the leading edge, because the pressure fluctuations due to nonlinear effects which furnish the strength of the trailing-edge noise are proportional to the square of the jet velocity. The trailing-edge noise is thus an order of magnitude smaller than that from the leading edge.

The issue of satisfying the Kutta condition at the supersonic trailing edge does not arise because the flow there is always regular, which makes it unnecessary to impose an eigensolution to the forced solution, as must be done for subsonic aerofoils (see, for example, Jones 1972; Crighton 1972, 1981, 1985). In potential theory for subsonic aerofoils, vortex shedding from a sharp trailing edge and energy exchange between acoustic waves and vortices is essential, because the solution to the inviscid forcing problem in that case is continuous but singular at the trailing edge. To satisfy the Kutta condition which requires finite behaviour at the sharp edge, a constant multiple of an eigensolution, also singular at the trailing edge, must be added to the forced solution. The multiplying constant can be chosen such that the singularity in the forced solution is exactly offset by that in the eigensolution. As a result, the solution becomes unique and regular, but the property of continuity is sacrificed, which indicates the formation of a vortex sheet. In the case of supersonic aerofoils, however, no such sacrifice is needed because the solution to the forcing problem is itself both continuous and regular. Of course, eigensolutions still exist for the supersonic problem, but none of them are physically acceptable. In this case, adding a constant multiple of an eigensolution to the forced solution yields a result that is discontinuous, singular and not unique, because there is now no criterion for the determination of the multiplying constant, unless it is chosen to be identically zero. Thus, the only acceptable solution is the forced solution itself.

The author would like to thank Professor J. E. Ffowcs Williams for many helpful discussions and comments.

Appendix

In this Appendix, the pressure fluctuations induced by a semi-infinite aerofoil cutting through a steady jet are derived in terms of an inverse Fourier transform. To distinguish from the case of finite-chord aerofoils, the pressure perturbation here is denoted by the capital letter $P(\boldsymbol{x}, t)$, which also satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \nabla^2 P = 0.$$

The Fourier transform of this equation yields the solution

$$\hat{P}(k_\alpha, k_3, \omega) = A(k_\alpha, \omega)\,\mathrm{e}^{\mathrm{i}\gamma x_3}, \tag{A 1}$$

which complies with the radiation condition at infinity if γ is chosen according to (2.4), where A is a constant to be determined from the boundary condition on the plane $x_3 = 0$.

In the present case of a semi-infinite aerofoil, it is convenient to specify the boundary condition in terms of the induced velocity fluctuations in the x_3 direction:

$$u_0 H(a^2 - x_\alpha^2)\,H(cMt - x_1). \tag{A 2}$$

This follows from the fact that the induced velocity vanishes ahead of the aerofoil, because no disturbance can travel faster than the supersonic leading edge, while on the aerofoil surface it also vanishes except in the region covered by the jet flow, in which it must be opposite to the jet velocity to comply with the boundary condition that the total normal velocity is zero. The Fourier transform of (A 2) can be found to be

$$\frac{2\pi \mathrm{i} a u_0}{\omega}\frac{J_1(a\lambda)}{\lambda}.$$

The velocity perturbation is related to the pressure fluctuations by the linear momentum equation which leads to the boundary condition for $\hat{P}$

$$\frac{\partial \hat{P}}{\partial x_3} = -2\pi \rho_0 a u_0 \frac{J_1(a\lambda)}{\lambda}.$$

The constant A in the result (A 1) can be determined from this, and the inverse Fourier transform gives

$$P(\boldsymbol{x}, t) = \frac{\mathrm{i}\rho_0 u_0 a}{(2\pi)^2} \int_{k_\alpha} \int_{k_0} \frac{J_1(a\lambda)}{\lambda} \frac{1}{\gamma(k_\alpha, k_0)} \mathrm{e}^{\mathrm{i}\gamma x_3} \mathrm{e}^{-\mathrm{i}(k_\alpha x_\alpha + k_0 cMt)} \mathrm{d}^2 k_\alpha \, \mathrm{d}k_0, \qquad \text{(A 3)}$$

from which the result (2.5) quoted in §2 follows immediately by setting $x_3 = 0$.

With γ defined by (2.4), it is necessary, when calculating the integrals in (A 3) in the complex wavenumber–frequency space, to choose the proper branch with branch cuts emanating from the two singular points determined by $\gamma = 0$. In doing so, both the radiation and the causality condition must be satisfied; the former requires the path of integration in the wavenumber plane, the k_1 plane for example, to be above any singularities in the left half-plane where the real part of k_1, $\mathrm{Re}(k_1)$, is negative, and below all in the region where $\mathrm{Re}\,(k_1)$ is positive. The causality condition, on the other hand, is met by chosing the path of integration in the frequency plane such that it runs above all the singularities (Morse & Feshbach 1953). From (A 3), it is also clear that the contour for the non-zero contribution to the k_0 integral must be closed by the semicircle at infinity in the lower half-plane so that the exponential factor in (A 3) gives a vanishingly small contribution on this semicircle. When the contour is closed in the upper half-plane, the integral is zero and corresponds trivially to the pressure fluctuation at $\boldsymbol{x}$ before the first wave generated arrives.

It should also be noted that the choice of the branch cuts is not completely arbitrary; whether the cuts should go to infinity or should join together at a finite point depends on whether infinity is a singular point. Take k_1 and k_0 for example. From the definition (2.4), we have

$$\gamma \rightarrow \mathrm{i}|k_1| \quad \text{as} \quad |k_1| \rightarrow \infty,$$
$$\gamma \rightarrow Mk_0 \quad \text{as} \quad |k_0| \rightarrow \infty.$$

Thus, infinity is a singular point in the k_1 plane but is not in the k_0 plane. The cuts in the k_1 plane must then be drawn from the two branch points to infinity with the path of integration running between them, a choice required to meet the radiation condition, while that in the k_0 plane should join the two branch points so that the path of interaction is above both of them, as is consistent with the casuality condition.

REFERENCES

Amiet, R. K. 1986*a* Airfoil gust response and the sound produced by aerofoil–vortex interaction. *J. Sound Vib.* **107**, 487–506.

Amiet, R. K. 1986*b* Intersection of a jet by an infinite span aerofoil. *J. Sound Vib.* **111**, 499–503.

Amiet, R. K. 1988 Comment on the surface pressure produced by an aerofoil–gust interaction. *J. Sound Vib.* **121**, 571–572.

Cargill, A. M. 1982 Low-frequency sound radiation and generation due to the interaction of unsteady flow with a jet pipe. *J. Fluid Mech.* **121**, 59–105.

Crighton, D. G. 1972 Radiation properties of the semi-infinite vortex sheet. *Proc. R. Soc. Lond.* A **330**, 185–198.

CRIGHTON, D. G. 1981 Acoustics as a branch of fluid mechanics. *J. Fluid Mech.* **106**, 261–298.

CRIGHTON, D. G. 1985 The Kutta condition in unsteady flow. *Ann. Rev. Fluid Mech.* **17**, 411–445.

FFOWCS WILLIAMS, J. E. & GUO, Y. P. 1988 Sound generated from the interruption of a steady flow by a supersonically moving aerofoil. *J. Fluid Mech.* **195**, 113–135.

FFOWCS WILLIAMS, J. E. & HALL, L. H. 1970 Aerodynamic sound generation by turbulent flow in the vicinity of a scattering half plane. *J. Fluid Mech.* **40**, 657–670.

FFOWCS WILLIAMS, J. E. & HAWKINGS, D. L. 1969 Sound generation by turbulence and surfaces in arbitrary motion. *Phil. Trans. R. Soc. Lond.* A **264**, 321–342.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1980 *Table of Integrals, Series and Products* (Corrected and enlarged edition). Academic.

GUO, Y. P. 1989 On sound generation by a jet flow passing a semi-infinite aerofoil. *AIAA Paper* 89–1070.

HOWE, M. S. 1976 The influence of vortex shedding on the generation of sound by convected turbulence. *J. Fluid Mech.* **76**, 711–740.

HOWE, M. S. 1980 The dissipation of sound at an edge. *J. Sound Vib.* **70**, 407–411.

HOWE, M. S. 1988 Contributions to the theory of sound production by vortex–aerofoil interactions, with applications to vortices with finite axial velocity defect. *Proc. R. Soc. Lond.* A **420**, 157–182.

JONES, D. S. 1972 Aerodynamic sound due to a source near a half plane. *J. Inst. Maths Applics.* **9**, 114–122.

JONES, D. S. 1982 *The Theory of Generalized Functions*. Cambridge University Press.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. I. General theory. *Proc. R. Soc. Lond.* A **211**, 1–32.

LIGHTHILL, M. J. 1978 *Waves in Fluids*. Cambridge University Press.

MILES, J. E. 1959 *The Potential Theory of Unsteady Supersonic Flow*. Cambridge University Press.

MORGAN, J. D. 1974 The interaction of sound with a semi-infinite vortex sheet. *Q. J. Mech. Appl. Maths* **27**, 465–487.

MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*. McGraw-Hill.

RIENSTRA, S. W. 1981 Sound diffraction at a trailing edge. *J. Fluid Mech.* **108**, 443–460.